Understanding and Developing Proofs with the Aid of Technology

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Abstract: *This paper continues the work entitled* On Proof Techniques and Technology *[2], which discusses strategies on how technology may be used to explore and understand mathematical ideas, and develop different kinds of proofs.*

1. Introduction

Proofs play an important part of mathematics. Although the function of proofs is traditionally viewed as mere verification, a proof has other functions as well. Proofs provide a means for giving insight on why something is true, for organizing thoughts into a system of ideas, for discovering new results, for communicating mathematical knowledge, and for challenging oneself [7]. Students of mathematics have to be able to read and comprehend proofs, as well as to construct them. The construction of good mathematical proofs can be difficult and challenging. Creating proofs is an art that improves with experience. In this paper, we highlight how technology helps students develop an in-depth understanding of mathematical concepts that should assist them to write correct proofs. Indeed, technology is able to engage the learner in explorations initially leading to conjecture and then to experience what it means to construct knowledge [11]. Most students become more motivated when they are able to construct figures in solving proof-problems. Writing a proof includes a clear statement of the conjecture, which allows for the connection between the hypothesis and the conclusion. Technology allows for experimentation with diagrams and tables to see how parts of the hypothesis are related. Further, to adapt to the Philippine setting where commercial technology may be inaccessible, we make use of free software packages whenever possible.

2. Using Technology to Find Patterns

Technology helps us find patterns so we can find conjectures easily. Therefore, we can focus on their proof. In abstract algebra, for example, students often find the study of groups and rings very challenging, given its abstract nature. In recent years, with the emergence of technology, algebraic concepts have become more accessible to students.

For example, a very helpful supplement to teaching abstract algebra is the interactive site created by Joseph Gallian [30]. The site is a supplement to [8], but can work hand-in-hand with other textbooks. The technology exercises are designed to help formulate and test conjectures. The following is an exercise from the site when studying group structures.

Let $U(n)$ denote the set of all positive integers less than n and relatively prime to n. Determine if $U(n)$ is cyclic. Run the program for n = 8, 16, 32, 64, and 128 and make a conjecture. Run the program for 3, 9, 27, 81, 343, 5, 25, 125, 7, 49, 11, and 121 and make a conjecture. Run the program for n = 12, 20, 28, 44, 52, 15, 21, 33, 39, 51, 57, 69, 35, 55, 65, and 85 and make a conjecture. Screenshots of this applet are shown in Figure 1.

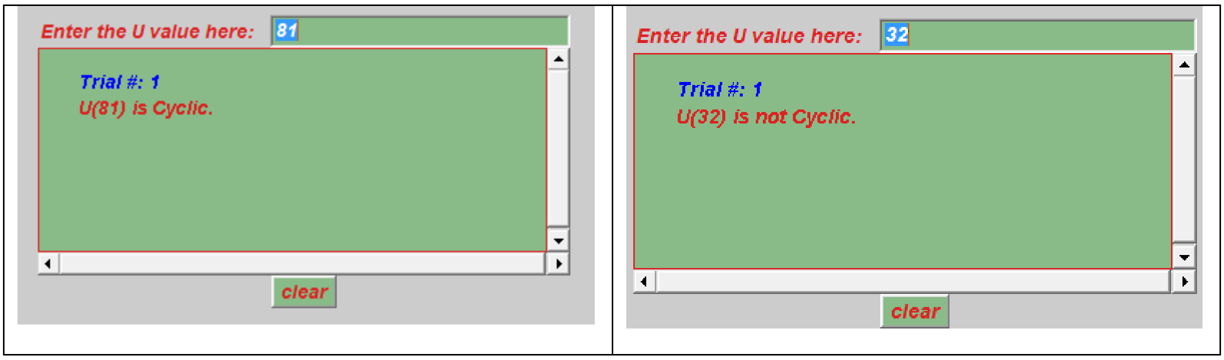


Figure 1: An abstract algebra applet (Output produced using [30])

Students may also find the color models and graphics generated by the abstract algebra feature of *Mathematica* [29] helpful in their investigations. The software package not only provides a means to study patterns, but also allows one to visualize abstract concepts. In ***cyclic.nb*** [18] (see Figure 2), we give outputs of the *Mathematica* package, particularly on the cyclic group Z_8. The students explore the generators of the cyclic group Z_n—the generators are elements a^k such that $(k,n) = 1$. Moreover, students can also visualize the construction of star polygons from generators of Z_n.

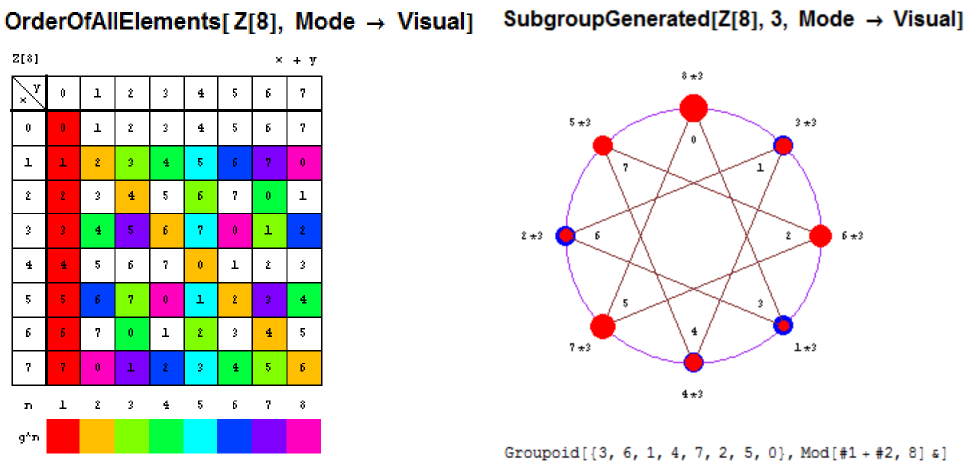


Figure 2: Color models and graphics (Output produced using *Mathematica version* 4.1 [29])

The Josephus problem is another example where technology can play a role in facilitating a proof. The problem is as follows. There are n people around a circle. Starting from person A, count k persons clockwise, and this k-th person is eliminated from the circle. This process continues until only one person is left. The problem is to determine the last remaining person, given values of n and k. Initially, students are tasked to determine patterns by simulating the problem using pen and paper. Students draw n shapes around a circle to represent people, and cross out every k-th shape to determine the last remaining person. However, this is a repetitive task. Once students have understood what the problem calls for, they may use technology, as provided by the Josephus web application [27]. By using this applet, students may see patterns right away. For example, for $k = 2$, students work out the following solution using the application.

Number of persons n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Last person remaining	1	1	3	1	3	5	7	1	3	5	7	9	11	13

Students may use this technology to input other values of n and k, and try to see whether they can see patterns as an initial step to the mathematical proof of their resulting conjecture.

3. Using Technology to Aid Discovery

The merit of proofs goes far beyond merely verifying results. Proofs open up new doors in mathematics, particularly those which serve as a means of inventing new results. With the aid of technology, one can gain insights and new discoveries pertaining to open problems in mathematics, leading to the development of proofs which support the solutions to the problems. Technology allows for the exploration and visualization of the various possible “solutions” of the problem. It also facilitates uncovering hidden contradictions and errors that produce counterexamples. In which case, mathematicians reconstruct old “proofs” and replace them with new ones.

As an illustrative example of this, the study of group structures in the hyperbolic plane and space remains an interesting problem for group theorists. Groups and their subgroups in Euclidean and spherical planes are relatively well known. However, those on the hyperbolic plane are somewhat less understood. This has to do with the immense variety of groups of hyperbolic isometries. In recent years, the opportunities to explore the non-Euclidean plane and space, and visualize the various mathematical properties that are satisfied in these planes, are available to mathematicians with the use of technology.

In [3, 4, 5, 10], the symmetry groups of semi-regular tilings on the hyperbolic plane are studied and characterized. An important tool in obtaining results on the symmetry groups was the software *Mathematica* using its *L2Primitive* and *Tess* add-on packages [9, 13]. The packages provide immediate access to different classes of tilings with varying properties and allow visualization of various symmetries present in tilings (see Figure 3). Another useful tool in studying semi-regular tilings is Don Hatch’s *Hyperbolic Tessellations Applet* [25] which tiles the hyperbolic plane via the Poincare disk model and allows navigation throughout the plane (see Figure 4). In this applet, the center of the tiling may be changed by simply dragging the desired area to the center of the disk.

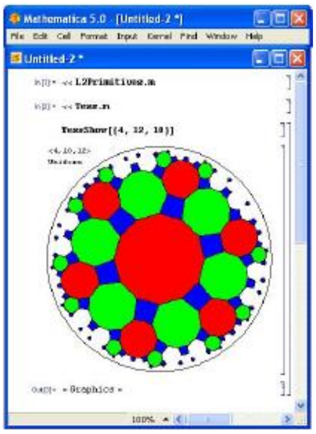


Figure 3: An output of a 4·12·10 tiling (Produced using Mathematica [29])

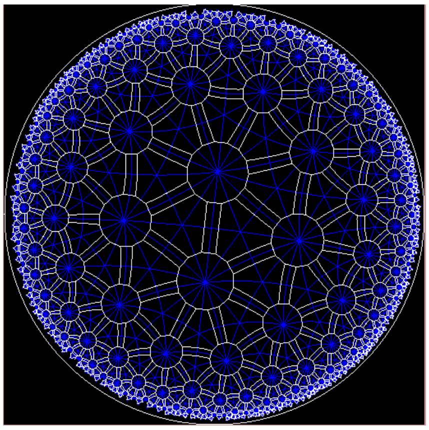


Figure 4: Hatch's Hyperbolic Tesselations Applet [25]

In the three-dimensional case, geometric properties, symmetry groups and subgroup structures of honeycombs in hyperbolic space, as well as Euclidean and spherical counterparts, may also be studied with technological tools. In [6], preliminary results on properties of the cubic honeycomb in the Euclidean space, as well as the subgroup structure of its associated symmetry group appeared. A useful tool in the study of properties of three-dimensional honeycombs includes Jeff Week's *Curved Spaces* [23] (see Figure 5). Curved Spaces is a flight simulator for multi-connected universes. This program allows the user to "fly though" spaces such as three-dimensional spherical, Euclidean, and hyperbolic space filled with certain polyhedra, allowing the study of various properties of honeycombs. On the other hand, the Magma program [1] allows the study of certain algebraic properties of these honeycombs, such as listing the subgroups of the corresponding symmetry group, up to a certain index.

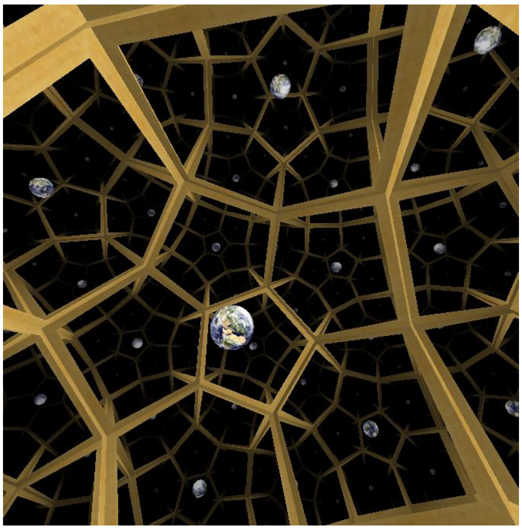


Figure 5: Week's Curved Spaces [23]

4. Using Technology to Explore Scenarios

Technology enables students to gain insight into a problem and explain why certain theorems are true. In the following example [12], we use the free dynamic mathematics software Geogebra [24], which operates on the Java [26] platform, to recreate the given problem. The problem is as follows.

> *A band of pirates buried their treasure on an island in the following manner. Along the shore were two large rocks. Somewhere between the rocks, but about 80 feet from the shore, was a large palm tree. One pirate stood at each rock and faced the palm tree. Each made a 90-degree turn and walked a distance equal to the distance from his rock to the palm tree. The other pirates buried the treasure midway between the places where the two pirates ended up* (see Figure 6). *Years later, some adventurers sought to find the treasure. Unfortunately, when they reached the island, they found the rocks but the palm tree was gone! All seemed lost until someone established where the treasure might be. How did that person do it?*

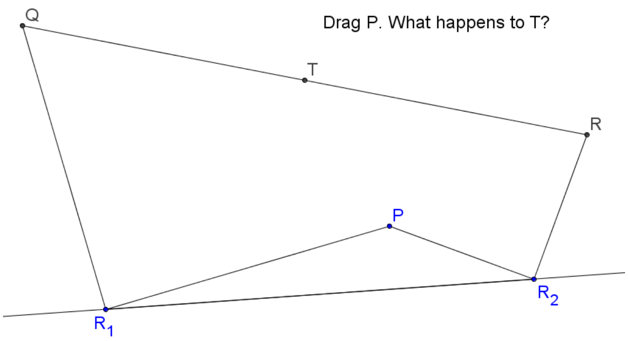


Figure 6: Buried treasure problem (Plot produced with Geogebra [24])

The dynamic geometry construction ***treasure.html*** [14] first allows students to verify that the location of the treasure T is independent of P, the location of the palm tree. Through independent explorations or guided exercises, students may perform additional constructions to get additional insight. In ***treasure2.html*** [15], several lines perpendicular to $\overline{R_1R_2}$ are constructed (see Figure 7). This allows students to see some congruent triangles which assist them in understanding the scenario, making a conjecture, and formulating their proof.

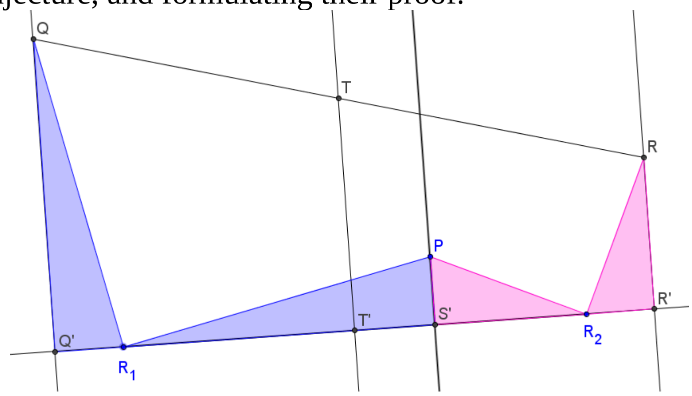
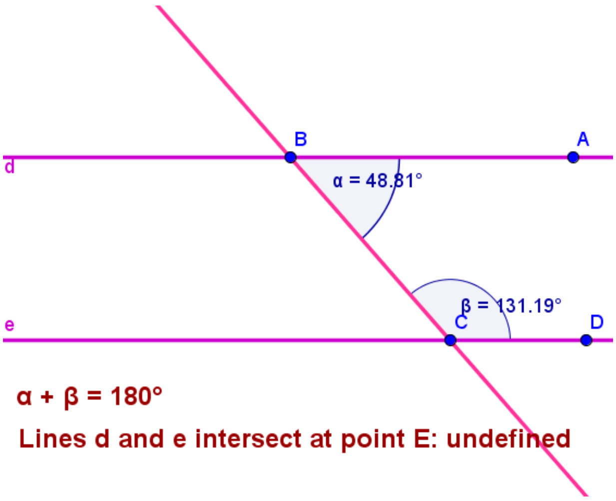
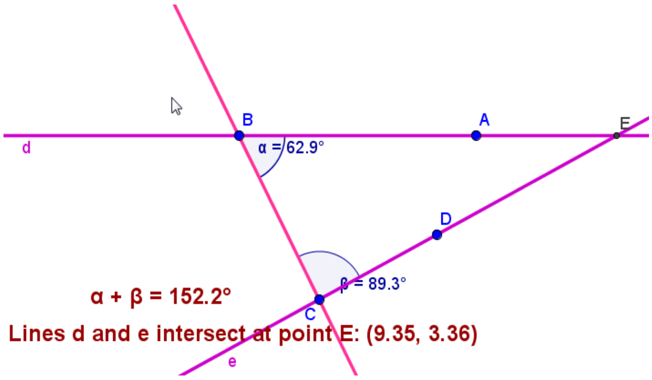


Figure 7: Additional constructions for the buried treasure problem (Plot produced with Geogebra [24])

A second example uses Dynamic Geometry Software to verify Euclid's Parallel Postulate: *If two lines are intersected by a transversal such that the sum of the degree measures of the interior angles on one side of a transversal is less than two right angles then the lines meet on that side of the transversal*. For instance, from ***euclids_fifth.html*** [19], students can make the following discoveries: If the sum of the degree measures of the interior angles α and β is 180°, then the lines *d* and *e* are parallel (Figure 8a). However, when point C is dragged, the sum of the degree measures of the angles changes. Consequently, students visualize the point of intersection E which eventuates when this happens, particularly on the side of the transversal where the sum of the degree measures of interior angles is less than 180° (Figure 8b).



(a)

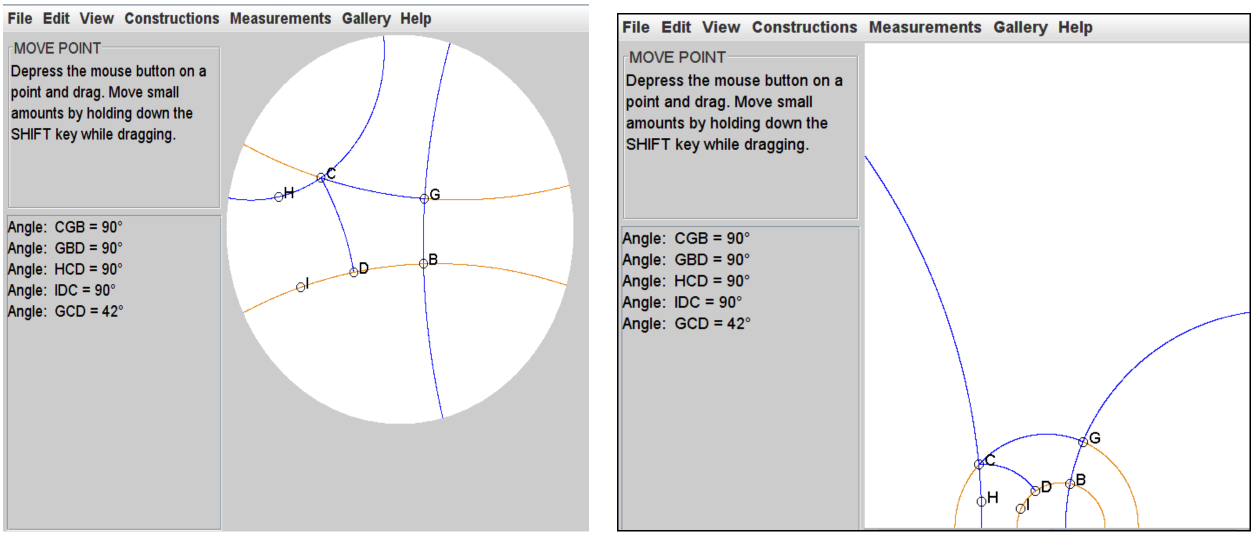


(b)

Figure 8: Euclid's Parallel Axiom (Plots produced with Geogebra [24])

As a follow-up activity, the students next explore Euclid's Parallel Postulate on non-Euclidean planes. Students are oftentimes challenged to imagine a universe beyond Euclid. With Dynamic Geometry, students can experience working in the hyperbolic plane and sphere, and explore the properties of the non-Euclidean geometries. In our teaching, we use the free software packages *Non-Euclid* [28] and *Spherical easel* [31]. The students explore the impossibility of Euclid's Parallel Postulate through its equivalent, Playfair's Axiom: *Given a line l and a point P not on the line, there is exactly one line through P parallel to l*. In the hyperbolic plane, the Hyperbolic Geometry Parallel Axiom states: *Given a line l and a point P not on l, there are at least two distinct lines which contain the point, and are parallel to l*. Whereas in the spherical plane, we have: *Given a line l and a point P not on l, there are no lines which can be drawn through the point, parallel to l*.

Using the file ***hyperbolic.euc*** [20], which opens with *Non-Euclid* [28], the student can work in the Poincare model as well as the upper half plane model. In Figures 9a and 9b, there are two lines through *C* parallel to line *DB*. Point *G* cannot be on *H*C, for if it were, then *GCDB* would be a rectangle. The measurement menu shows *HC* and *CG* parallel to *DB*.



(a) (b)

Figure 9: Hyperbolic geometry parallel axiom (Plot produced with Non-Euclid [28])

On the other hand, using the file ***spherical.ssp*** [21], which opens with *Spherical Easel* [31], the student can verify that given a line *l* and a point *P* not on *l*, no lines can be drawn through *P* parallel to *l*. In Figure 10, we have lines *m* and *n* passing through *P* intersecting *l* at *C* and *E*, respectively.

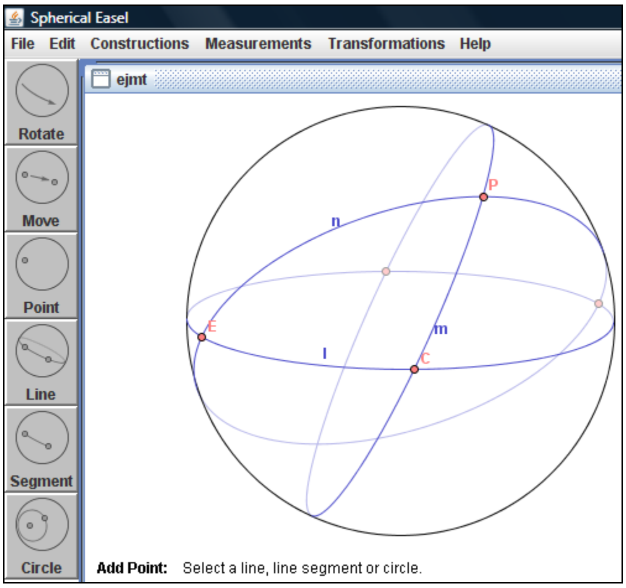
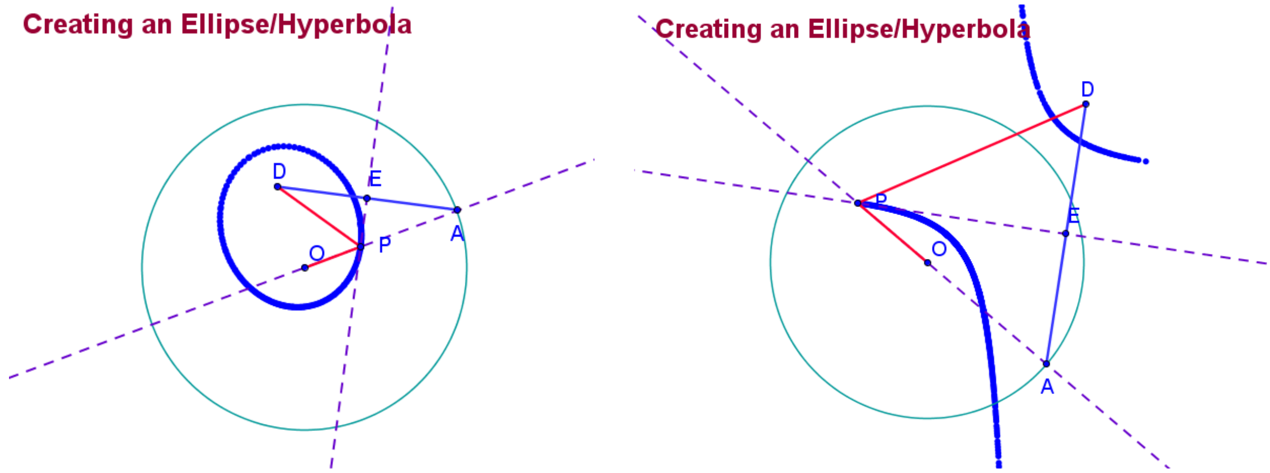


Figure 10: Spherical geometry parallel axiom
(Plot produced with Spherical Easel [31])

5. Using Technology to Aid Investigations

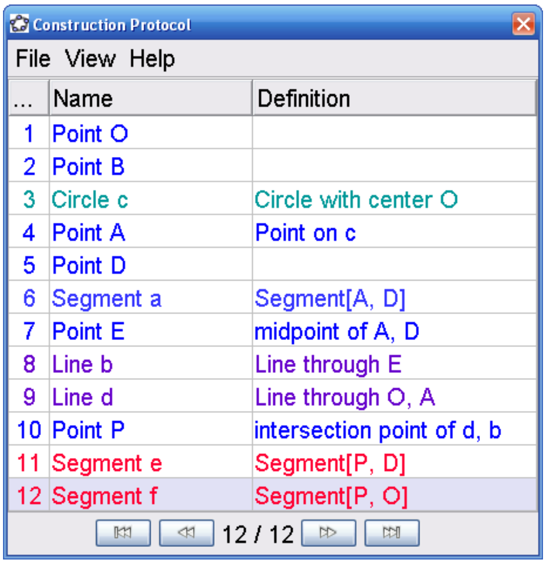
In an investigation, unlike a problem, it is not necessary that there is a specific question. Students experiment, come up with some questions of their own, and try to define conjectures. Students then attempt to verify whether the conjectures are true, and subsequently prove or disprove them.

For example, teachers may provide students with the pre-constructed Geogebra file ***creating_ellipse.html*** [22]. Then students can experiment on the consequences of the construction—for instance, by dragging the point *A* along the circle, they can observe the locus of points being formed (Figure 11a). They can also conjecture what happens to the set of points when point *D* is dragged outside the circle (see Figure 11b). Teachers can guide students to discover constant measurements, and subsequently define the ellipse and hyperbola. A very helpful feature of Geogebra is its capability to list the construction steps (see Figure 12). Using this feature as a guide, students are then tasked to justify these invariant measurements. For instance, in Figure 11a, students may observe that the lengths of segments *DP* and *AP* are equal, implying that the sum of the lengths of *DP* and *OP* is constant, and refer to the construction protocol in formulating their mathematical proof.



(a) (b)

Figure 11: Construction of an ellipse and hyperbola (Plot produced using Geogebra [24])



...	Name	Definition
1	Point O	
2	Point B	
3	Circle c	Circle with center O
4	Point A	Point on c
5	Point D	
6	Segment a	Segment[A, D]
7	Point E	midpoint of A, D
8	Line b	Line through E
9	Line d	Line through O, A
10	Point P	intersection point of d, b
11	Segment e	Segment[P, D]
12	Segment f	Segment[P, O]

Figure 12: Construction protocol (Output produced using Geogebra [24])

As a second example, students are asked to open ***conic.html*** [16]. Here, they are asked to use the trace function of Geogebra to formulate some conjectures about conics. Given a conic, and two points on the conic, they first construct the segment joining these two points. They then construct the midpoint of the segment and turn on the trace for this midpoint. Students are first encouraged to engage with the construction focusing especially on the path of the midpoint. But without specific tasks set, students may not observe anything right away (Figure 13).

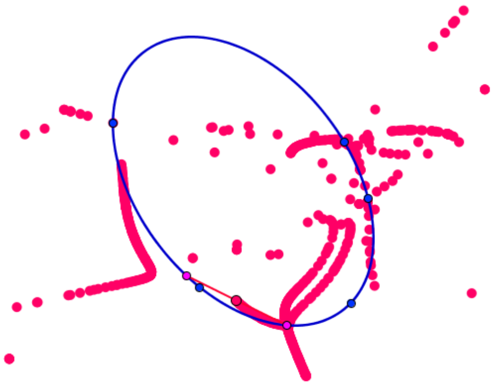


Figure 13: An investigation may initially not yield clean observations
(Plot produced using Geogebra [24])

However, if students drag one endpoint of the segment, they may observe several things. The locus of the midpoint appears to form a conic that is smaller than the original one (see Figure 14).

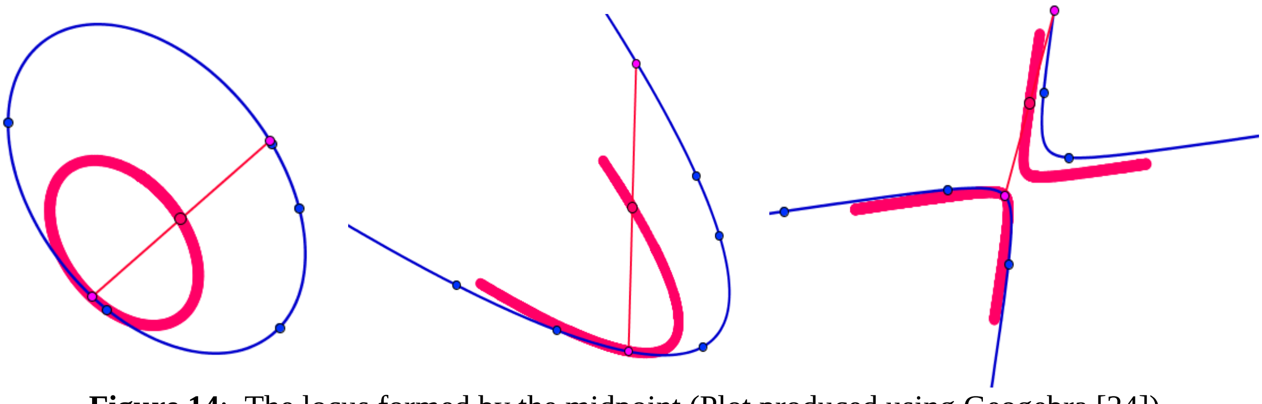


Figure 14: The locus formed by the midpoint (Plot produced using Geogebra [24])

A relevant question to ask is: Are the loci conics as well? Students use what they know about these conics to formulate their proof. They may even proceed to determine the relative sizes of the loci as compared to the original conics. As an extension, students may use ***conic2.html*** [17] to determine the loci of tracing an arbitrary fixed point on the segment (see Figure 15).

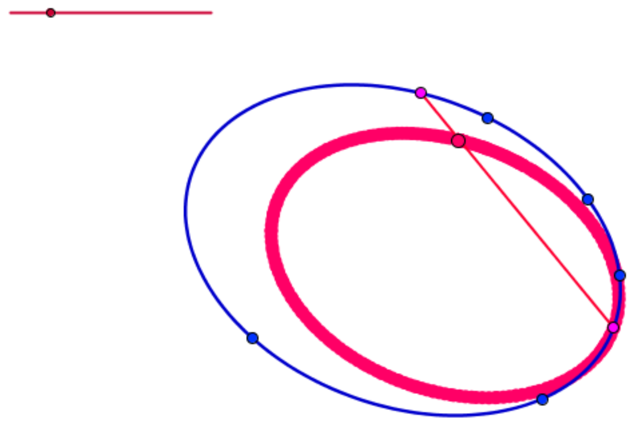


Figure 15: The locus formed by an arbitrary fixed point on the segment (Plot produced using Geogebra [24])

6. Concluding Remarks

Proofs are the heart of mathematics. They verify the validity of existing mathematical statements and provide a means of discovering and inventing new mathematical results. One of the challenges that confront mathematics students is being able to understand and construct good mathematical proofs. One of our roles as teachers is to inspire our students to study and appreciate the meaning of proofs in existing and new mathematical theories. We aim to develop their confidence to be able to create elegant and well-structured proofs. The emergence of technological tools, and technology support, such as software packages and interactive sites in recent years has made mathematics accessible and more enjoyable for students, even in the study and formulation of mathematical proofs. The focus of this paper is to emphasize some areas where students can be engaged in various levels of "proving with technology", arising from experiences gained in our classes. We would like to mention that experimentation and explorations with the aid of technology may not necessarily provide students with the ability to formulate an appropriate conjecture or to generate a suitable proof. These capabilities depend on how deeply the students can analyze and bring forward thoughts and insights, and on the ability of each individual package to demonstrate and display every applicable case. Definitely, these capabilities will improve with time, as the student develops his mathematical maturity.

There is still a lot to be studied in terms of strategies of proving with technology and a direction for future work would be to study its usefulness on student learning.

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